The cohomology with local coefficients of compact hyperbolic manifolds - long version

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Dedicated to M.S Raghunathan

Abstract

We extend the techniques developed by Millson and Raghunathan in [MR] to prove nonvanishing results for the cohomology of compact arithmetic quotients M of hyperbolic n-space $\mathbb{H}^n$ with values in the local coefficient systems associated to finite dimensional irreducible representations of the group $SO(n, 1)$. We prove that all possible nonvanishing results compatible with the vanishing theorems of [VZ] can be realized by sufficiently deep congruence subgroups of the standard cocompact arithmetic examples.

1   Introduction

The purpose of this paper is to extend the techniques of [MR] to prove nonvanishing results for the cohomology of certain arithmetic quotients M of hyperbolic n-space $\mathbb{H}^n$ with values in the local coefficient systems associated to a finite dimensional representations W of the group $SO(n, 1)$. In the compact case we replace the technique of intersecting pairs of totally-geodesic submanifolds considered there by intersecting the same submanifolds but now each is equipped with a *local coefficient*, i.e. a parallel section of the restriction of the associated local coefficient system $\widetilde{W}$ restricted to the cycle (or equivalently a nonzero vector in W fixed under the fundamental group of the submanifold). In §3.3.1 we explain how such cycles correspond to an especially simple class of cycles in the Eilenberg MacLane complex $C_\cdot(\Gamma) \otimes W$.

To define an intersection pairing of such cycles one needs a pairing on the local coefficients. The key fact that it can be arranged that the two manifolds intersect in a single connected component we borrow from the earlier papers [MR], [JM] and [FOR]. The remaining problems are to find for which W the required local coefficients exist and then to verify that the coefficient pairings applied to the local coefficients are nonzero.

In order to state our main theorem we need a definition.

Definition 1.1 *Let $\mu = (b_1, b_2, \cdots, b_m)$ be a dominant weight for $SO(n, 1)$ where $m = [\frac{n+1}{2}]$ and we use the usual Cartesian coordinates on the dual of the Cartan (so the positive roots are the sums and differences of the coordinate functionals in case $n+1$ is even and the coordinate functionals are the short positive roots if $n+1$ is odd). Then we define $i(\mu)$ to be the number of nonzero entries in μ.*

In what follows we will let Γ denote a member of the special family of cocompact torsion-free lattices of $SO(n, 1)$ which are congruence subgroups of some level $\mathfrak{b}$ of the group of units of the form $f(x)$ in $n+1$ real variables given by
$$f(x) = x_1^2 + \cdots + x_n^2 - \sqrt{m} \, x_{n+1}^2$$
for some square-free positive integer m and a sufficiently large ideal $\mathfrak{b}$ in the integers of $\mathbb{Q}(\sqrt{m})$. We will let Φ denote a general cocompact torsion-free lattice of $SO(n, 1)$.

Theorem 1.2 *Let W be the irreducible representation of $SO(n, 1)$ with highest weight μ. Then we have*

1. *If $n = 2m - 1$ and all entries of μ are nonzero then $H^p(\Phi, W) = 0$ for all cocompact lattices Φ of $SO(n, 1)$ and all p.*

2. *For all other W we have*

$$p \notin \{i(\mu), i(\mu) + 1, \cdots, n - i(\mu)\} \Rightarrow H^p(\Phi, W) = 0.$$

3. *For all other W we have (for $\mathfrak{b}$ sufficiently large depending on W)*

$$p \in \{i(\mu), i(\mu) + 1, \cdots, n - i(\mu)\} \Rightarrow H^p(\Gamma, W) \neq 0.$$

Remark 1.3 *The vanishing results are (essentially) due to Vogan and Zuckerman,[VZ]. A careful statement (without computational details) of what [VZ] implies for $SO(n, 1)$ can be found in [RS], §1.3. Roughly half the above nonvanishing results for the cocompact case were proved by J.- S. Li in [Li]. The rest are new as is the formulation of the theorem. See the end of the Introduction for a more complete discussion including a discussion of related results in the noncocompact case. If we consider the more general case of lattices Φ in $Spin(n, 1)$ and representations of $Spin(n, 1)$ the analogue of the above statements 1. and 2. above still hold by [VZ]. As for statement 3. one necessarily has $i(\mu) = m$ for otherwise one does not have a genuine*

representation of $Spin(n, 1)$ and the only case of interest is $n = 2m$. Let Φ be a torsion-free cocompact lattice in $Spin(2m, 1)$ and W have highest weight μ satisfying $i(\mu) = m$. Then one obtains $H^m(\Phi, W) \neq 0$ by the usual Euler characteristic argument.

We can also prove nonvanishing results for cup-products of cohomology groups with local coefficients. We give only one such example here.

The cohomology algebra with trivial coefficients $H^*(\Gamma, \mathbb{R})$ acts on the cohomology groups with local coefficients. We remind the reader that it was proved in [MR] that for the above cocompact lattices

$$H^p(\Gamma, \mathbb{R}) \neq 0, 0 \leq p \leq n.$$

In the last section of this paper we will prove a general result which implies

Theorem 1.4 *Suppose W is an irreducible finite-dimensional representation of $SO(n, 1)$ and $H^q(\Gamma, W) \neq 0$. Suppose $p \geq 1$ and that $p + q \leq [\frac{n}{2}]$. Then the cup-product*

$$H^p(\Gamma, \mathbb{R}) \otimes H^q(\Gamma, W) \to H^{p+q}(\Gamma, W)$$

is a nonzero map.

Remark 1.5 *We can use the previous theorem to explain why the set of positive integers k such that $H^k(\Gamma, W) \neq 0$ is an unbroken string of integers. Indeed by operating on $H^*(\Gamma, W)$ by the cohomology with trivial coefficients and applying the previous theorem we have $H^{i(\mu)}(\Gamma, W) \neq 0 \implies H^k(\Gamma, W) \neq 0, i(\mu) \leq k \leq [\frac{n}{2}]$. But then by Poincaré duality we have $H^k(\Gamma, W) \neq 0, i(\mu) \leq k \leq n - i(\mu)$.*

It is important to observe that there is considerable overlap between this theorem and the work of Jian-Shu Li, [Li]. In [Li], Professor Li proves nonvanishing results for cohomology groups with local coefficients for cocompact lattices in the classical groups $SO(p, q), Sp(p, q), SU(p, q), Sp_{2m}(\mathbb{R})$ and $SO^*(2n)$. When specialized to the case of $SO(n, 1)$ his results give roughly half of the nonvanishing results in our Theorem 1.2 in that he proves nonvanishing for those p in the range $i(\mu) \leq p < \frac{n-1}{4}$ (and the Poincarë dual dimensions). We also note that the special case $H^1(\Phi, W) \neq 0 \Leftrightarrow i(\mu) = 1$ of the above theorem is a consquence of the results of [Ra] and [M2]. The result that $i(\mu) > 1 \Rightarrow H^1(\Phi, W) = 0$ for all Φ is a special case of Raghunathan's Vanishing Theorem, [Ra] and the result that $i(\mu) = 1 \Rightarrow H^1(\Gamma, W) \neq 0$ was proved in [M2].

We also need to mention the large amount of work on the noncompact case. In this case there is a large supply of cohomology coming from the Borel- Serre boundary, [BS]. The idea of using Eisenstein series to promote boundary classes to classes inside began with the fundamental papers of Harder [Ha1] and [Ha2] and nonvanishing theorems for this case analogous to the one we prove for the compact case were proved by Harder, loc. cit.,

Rohlfs and Speh, [RS] and Speh, [S]. For the noncompact case one has the harder problem of finding cuspidal classes. Progess on this for the case treated in this paper is to be found in [RS]. More general results that have some application to the case considered here can be found in [BLS],[LS] and [C]. It is possible to use the intersection-theoretic methods of this paper to prove nonvanishing results for the finite-volume noncompact case. We will discuss this in later work.

In joint work with Jens Funke, [FM2], we will show that the results of [KuM] concerning the connection between the theta lifting and cycles in cocompact quotients of the symmetric spaces of $SO(p,q)$ and $SU(p,q)$ generalize to local coefficients. In particular the Poincaré dual harmonic forms of some of the cycles with local coefficients considered here can be obtained from an appropriate theta-lifting of Siegel modular forms of genus equal to the codimension of the cycles.

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This paper is dedicated to M.S.Raghunathan on the occasion of his sixtieth birthday. It is different from the one that I presented at the conference at the Tata Institute in his honor in December of 2001. It was felt that paper was too long for the conference volume and it appeared difficult to split it up in a satisfactory fashion. However this paper is perhaps more appropriate in that it depends in an essential way on the techniques Raghunathan and I developed in [MR]. It is my pleasure to acknowledge the great impact that Raghunathan had at the beginning of my career, by providing a critical insight in [M1] and by the collaboration [MR].

2   The vanishing theorem

The goal of this section is to prove the following theorem. Let $m = [\frac{n+1}{2}]$. Let W be the irreducible representation of $Spin(n,1)$ with highest weight $\mu = (b_1, b_2, \cdots, b_m)$. Recall that $i(\mu)$ is the number of nonzero entries of μ.

Theorem 2.1 *Let Φ be a cocompact lattice in $SO(n,1)$. Then*

1. If n is odd and $i(\mu) = l$ then

$$H^p(\Phi, W) = 0, \text{ for all } p.$$

2. In all other cases

$$H^p(\Phi, W) = 0, p \leq i(\mu) - 1.$$

The Theorem 2.1 follows by working out what the results of Vogan and Zuckerman in [VZ] imply for the case in hand. We will establish some notation and then review the theory of Vogan and Zuckerman.

First we will use the following convention to deal with the problems of denoting a Lie group or Lie algebra and its complexification. We will use Roman letters e.g. G, K, L to denote real Lie groups. We will use the corresponding gothic letters with a subscript 0 e.g $\mathfrak{g}_0, \mathfrak{k}_0, \mathfrak{l}_0$ to denote their Lie algebras and the same symbol without the subscript 0 to denote the complexification of the corresponding real Lie algebra. Furthermore, a gothic letter without a subscript 0 e.g. $\mathfrak{q}$ or $\mathfrak{u}$ will denote a *complex* Lie algebra.

We let V be an $n + 1$-dimensional real vector space equipped with a symmetric bilinear form of signature $(n, 1)$. We choose a basis $\{e_1, \cdots, e_{n+1}\}$ such that $(e_i, e_j) = 0, i \neq j$, $(e_i, e_i) = 1, 1 \leq i \leq n$ and $(e_{n+1}, e_{n+1}) = -1$. We will need to use a Witt basis for the complexification $V \otimes \mathbb{C}$.

In case $n + 1$ is odd put $n = 2m$. Define $u_i = e_i - \imath e_{m+i}$ and $v_i = e_i + \imath e_{m+i}, 1 \leq i \leq m$. We arrange the basis in the order $\{u_1, \cdots, u_m, e_{n+1}, v_m, \cdots, v_1\}$.

In case $n + 1$ is even, put $n + 1 = 2m$. Define u_i and v_i for $1 \leq i \leq m - 1$ as before. Define u_m and v_m by $u_m = e_m - e_{2m}$ and $v_m = e_m + e_{2m}$. We then define our ordered basis to be $\{u_1, \cdots, u_m, v_m, \cdots, v_1\}$.

In both cases we will call the above basis a Witt basis (even though $(u_i, v_i) = 2, 1 \leq i \leq m$).

We define a Cartan involution $\theta : V \to V$ by

$$\theta(e_i) = \begin{cases} e_i, & 1 \leq i \leq n \\ -e_i, & i = n + 1. \end{cases}$$

The centralizer of θ, to be denoted K, is then a maximal compact subgroup isomorphic to $O(n)$. We will let K^0 denote the connected component of the identity of K. We let o be the point in $\mathbb{H}^n$ fixed by K.

We let $\mathfrak{g}_0$ denote the Lie algebra of $G = SO(n, 1)$ and $\mathfrak{k}_0$ denote the Lie algebra of K^0. We let $\mathfrak{p}_0$ be the orthogonal complement to $\mathfrak{k}_0$ for the Killing form. Thus we have a canonical isomorphism

$$\mathfrak{p}_0 \cong T_o(\mathbb{H}^n).$$

We represent $\mathfrak{g}$ as $n + 1$ by $n + 1$ matrices using the Witt basis. We let $\mathfrak{h}$ be the Cartan subalgebra of $\mathfrak{g}$ consisting of diagonal $n + 1$ by $n + 1$ matrices with diagonal entries $(x_1, \cdots, x_l, -x_l, \cdots, -x_1)$ in case $n + 1$ is even and $(x_1, \cdots, x_l, 0, -x_l, \cdots, -x_1)$ in case $n + 1$ is odd.

In what follows it will be convenient to use the canonical isomorphism $\phi : \bigwedge^2 V \to \mathfrak{so}(V)$ for V a (real or complex) vector space equipped with a nondegenerate symmetric bilinear form (,). The isomorphism is determined by the formula

$$\phi(x \wedge y)(z) = (x, z)y - (y, z)x.$$

We will use ϕ to identify $\bigwedge^2 V$ and $\mathfrak{so}(V)$ henceforth.

We have

$$\mathfrak{h} = \oplus_i^m \mathbb{C} \, u_i \wedge v_i.$$

In the case that $n+1$ is odd we have $\mathfrak{h} \subset \mathfrak{k}$. In the case that $n+1$ is even we have

$$\mathfrak{h} \cap \mathfrak{p} = \mathbb{C}\, u_m \wedge v_m = \mathbb{C}\, e_m \wedge e_{2m} = \mathfrak{a}.$$

Here $\mathfrak{a}$ is the complexification of the Cartan subspace $\mathfrak{a}_0 = \mathbb{R}\, e_m \wedge e_{2m}$ of $\mathfrak{p}_0$. We set $\mathfrak{t} := \mathfrak{h} \cap \mathfrak{k}$ whence (if $n+1$ is even)

$$\mathfrak{h} = \mathfrak{t} \oplus \mathfrak{a}.$$

2.1 The representations $A_{\mathfrak{q}}(\lambda)$.

In [VZ] the authors constructed all unitary representations π of a semisimple Lie group G with nonzero continuous cohomology in a finite dimensional representation W of G. These representations are parametrized by pairs $(\mathfrak{q}, \lambda)$ consisting of a θ–stable parabolic subalgebra $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$ and a unitary one-dimension representation λ of L. They will be denoted $A_{\mathfrak{q}}(\lambda)$. We will also use λ to denote the corresponding representation of $\mathfrak{l}_0$. In what follows we let L^c be the compact form of the complexification $L_{\mathbb{C}}$ of L.

Let $R = dim(\mathfrak{u} \cap \mathfrak{p})$ and $S = dim(\mathfrak{u} \cap \mathfrak{k})$. The following fundamental lemma is proved in [BW], the first statement is Theorem 5.2, pg. 222 and the second (Wigner's Lemma) is Corollary 4.2, pg.26.

Theorem 2.2 *In order that there exist a cocompact lattice $\Gamma \subset G$ such that $H^p(\Gamma, W) \neq 0$ it is necessary that there exists an irreducible unitary representation π such that the*

$$H^p_{cont}(G, \pi \otimes W^*) \neq 0.$$

Furthermore the nonvanishing of $H^p_{cont}(G, \pi \otimes W^*) \neq 0$ implies that the infinitesimal characters of π and W coincide.

To obtain Theorem 2.1 we have only to examine which representations $A_{\mathfrak{q}}(\lambda)$ have the same infinitesimal character as that of W and calculate the lowest dimensions in which they have continuuous cohomology.

We have

Lemma 2.3   *1. The infinitesimal character of $A_{\mathfrak{q}}(\lambda)$ is $\lambda + \rho(\mathfrak{g})$*

2. *Let W be the finite dimensional representation with infinitesimal character $\lambda + \rho(\mathfrak{g})$ (so W has highest weight λ). Then the representation $A_{\mathfrak{q}}(\lambda) \otimes W^*$ has cohomology degrees $R + j$ for those j such that $H^j(L^c/L^c \cap K, \mathbb{R}) \neq 0$.*

3. *In particular the only finite dimensional representation W for which $H^k(A_{\mathfrak{q}}(\lambda) \otimes W^*) \neq 0$ for any k is the representation W_λ with highest weight λ and the lowest degree k in which $A_{\mathfrak{q}}(\lambda) \otimes W_\lambda^*$ can have nonzero cohomology is in degree R.*

Proof. The first statement is (b) of Theorem 5.3 of [VZ] and the second in contained in Theorem 5.5 of [VZ]. $\qquad\square$

Let $\Delta \in \mathfrak{h}$ be the dual cone to the cone of positive roots and $x \in \Delta$. We recall the definition of the standard parabolic subalgebra $\mathfrak{q}(x)$ associated to x. We have

$$\mathfrak{q}(x) = \mathfrak{l}(x) \oplus \mathfrak{u}(x)$$
$$\text{with}$$
$$\mathfrak{l}(x) = \mathfrak{h} \oplus \sum_{\langle \alpha, x \rangle = 0} \mathfrak{g}_\alpha \text{ and } \mathfrak{u}(x) = \sum_{\langle \alpha, x \rangle > 0} \mathfrak{g}_\alpha$$

We also define the standard parabolic $\mathfrak{q}(\lambda)$ associated to λ in the weight cone (so $\lambda \in \mathfrak{h}^*$ and $\langle \lambda, \alpha^\vee \rangle \geq 0$ for all positive coroots $\alpha^\vee$). The algebra is defined in the same way as $\mathfrak{q}(x)$ be replacing x by λ and $\langle \ , \ \rangle$ by the (dual) Killing form $(\ , \ )$.

In what follows we will let $\Delta(x)$, resp. $\Delta(\lambda)$, denote the subset of the *positive* roots that are orthogonal to x, resp. λ.

Definition 2.4 *We will say a character $\lambda \in \mathfrak{h}^*$ is compatible with a parabolic subalgebra $\mathfrak{q}$ if λ extends to a character of $\mathfrak{l}$ or equivalently if λ annihilates the commutator $[\mathfrak{l}, \mathfrak{l}]$.*

Note that the definition of the representation $A_\mathfrak{q}(\lambda)$ assumes that $\mathfrak{q}$ and λ are compatible.

We have

Lemma 2.5 *If λ is compatible with a standard parabolic subalgebra $\mathfrak{q}(x)$ then*

$$R(\mathfrak{q}(x)) = dim(\mathfrak{u}(x) \cap \mathfrak{p}) \geq dim(\mathfrak{u}(\lambda) \cap \mathfrak{p}) = R(\mathfrak{q}(\lambda)).$$

Proof. We claim that $\Delta(x) \subset \Delta(\lambda)$. Indeed suppose that $\alpha \in \Delta(x)$. Then $h_\alpha = [x_\alpha, y_\alpha]$ with $x_\alpha, y_\alpha \in \mathfrak{l}(x)$. Hence $h_\alpha \in [\mathfrak{l}(x), \mathfrak{l}(x)]$. Since by assumption λ annihilates this commutator subalgebra we have $\lambda(h_\alpha) = 0 \Rightarrow (\lambda, \alpha) = 0 \Rightarrow \alpha \in \Delta(\lambda)$. The claim follows. Thus $\Delta(\lambda)^c \subset \Delta(x)^c$. Equivalently, for a positive root β we have $(\beta, \lambda) > 0 \Rightarrow \langle \beta, x \rangle > 0$ and consequently

$$\mathfrak{u}(\lambda) \subset \mathfrak{u}(x)$$

and the lemma follows. $\qquad\square$

We will first deal with the case of $SO(2m, 1)$.

2.2 The case of $SO(2m, 1)$

We have $l = m$. Now suppose we are given W with highest weight $(b_1, \cdots, b_m)$. We assume $i(\lambda) = r$ and we set $s = m - r$.

Theorem 2.1 for the case $n = 2m$ will follow from

Lemma 2.6 *Suppose that λ is as above. Then we have*

1. *$L(\lambda) = C \times SO(2s, 1)$ where $C \subset K$.*

2. *The set $\{u_i \wedge e_{2m+1} : 1 \leq i \leq r\}$ is a weight basis for $\mathfrak{u}(\lambda) \cap \mathfrak{p}$. The vector $u_i \wedge e_{2m+1}$ has weight $\epsilon_i, 1 \leq i \leq r$.*

3. *We have*
$$R(\mathfrak{q}(\lambda)) = i(\lambda)$$

4. *Let W be the finite-dimensional representation with highest weight λ. Then the representation $A_{\mathfrak{q}(\lambda)}(\lambda) \otimes W^*$ has cohomology in degrees $i(\lambda)$ and $n - i(\lambda)$.*

Proof.

We will prove only the last statement. We note that $L = C \times SO(2s, 1)$ with $s = m - i(\lambda)$. Hence $L^c = C \times SO(2s + 1)$ and $L^c/L^c \cap K = S^{2s}$. Thus $A_{\mathfrak{q}}(\lambda) \otimes W^*$ has nonzero cohomology in degrees $i(\lambda)$ and $i(\lambda) + (2(m - i(\lambda))) = 2m - i(\lambda) = n - i(\lambda)$. $\square$

We can now prove Theorem 2.1 for the case $n = 2m$. Note that in the statement of the next theorem we are not assuming that $\mathfrak{q} = \mathfrak{q}(\lambda)$ only that $\mathfrak{q}$ and λ are *compatible*.

Theorem 2.7
$$H^j(A_{\mathfrak{q}}(\lambda) \otimes W_\lambda^*) = 0, \; for \; j < i(\lambda).$$

Proof. We claim we have
$$i(\lambda) \leq R(\mathfrak{q}).$$

We deduce the inequality from the equality $i(\lambda) = R(\mathfrak{q}(\lambda))$ (Lemma 2.6) and the inequality $R(\mathfrak{q}) \geq R(\mathfrak{q}(\lambda))$ from Lemma 2.5. The claim follows. Hence $j < i(\lambda) \Rightarrow j < R(\mathfrak{q}) \Rightarrow H^j(A_{\mathfrak{q}}(\lambda) \otimes W_\lambda^*) = 0$ by Lemma 2.3. $\square$

2.3 The case of $SO(2m - 1, 1)$

For the case of $SO(2m - 1, 1)$ we reason as above. Again we have $l = m$. However Lemma 2.6 must be modified in case $i(\lambda) = m$.

Lemma 2.8 *Suppose that $i(\lambda) = m$. Then there does not exist $\mathfrak{q}$ such that λ is compatible with $\mathfrak{q}$ and $\lambda|\mathfrak{l}$ is infinitesimally unitary.*

Proof. By assumption $\mathfrak{h} \subset \mathfrak{l}$ for any standard $\mathfrak{q}$. But we have $\mathfrak{a} \subset \mathfrak{h}$. Now if $\eta = (a_1, \cdots, a_m) \in \mathfrak{h}^*$ is infinitesimally unitary then its last component a_m must be pure imaginary. But the last component of λ is a *nonzero* integer by assumption. $\square$

Thus we obtain

Theorem 2.9 *Let W be a finite-dimensional irreducible representation of $SO(2m-1,1)$ with highest weight λ satisfying $i(\lambda) = m$. Then for all p and all cocompact lattices $\Phi \subset SO(n,1)$ we have*

$$H^p(\Phi, W) = 0.$$

The case in which $i(\lambda) < m$ is roughly the same as for the case of $SO(2m,1)$ and is left to the reader. However we caution the reader that in this case the space $\mathfrak{u} \cap \mathfrak{p}$ is not $\mathfrak{h}$-stable (because it is not $\mathfrak{a}$-stable),

In particular we have

Lemma 2.10   *1. $L(\lambda) = C \times SO(2s-1,1)$ where $C \subset K$.*

2. The set $\{u_i \wedge e_{2m} : 1 \leq i \leq i(\lambda)\}$ is a basis (but not a weight basis) for $\mathfrak{u}(\lambda) \cap \mathfrak{p}$.

3. We have
$$R(\mathfrak{q}(\lambda)) = i(\lambda)$$

4. Let W be the finite-dimensional representation with highest weight λ. Then the representation $A_{\mathfrak{q}}(\lambda) \otimes W^*$ has cohomology in degrees $i(\lambda)$ and $n - i(\lambda)$.

Theorem 2.1 for the case of $SO(2m-1,1)$ and $i(\lambda) < m$ now follows from the previous lemma in the same way as for the case of $SO(2m,1)$.

3   The intersection theory of cycles with local coefficients

3.1   Simplicial homology and cohomology with local coefficients

3.1.1   The definition of the groups

We begin by recalling that the fundamental groupoid of a connected topological space X is the category whose objects are the points of X and such that for $x, y \in X$ the morphisms, $Mor(x, y)$ from x to y are the homotopy classes of paths from x to y.

Definition 3.1 *A coefficient system (or local coefficient system or local system) $\mathcal{G}$ of abelian groups over a topological space X is a covariant functor from the fundamental groupoid of X to the category of abelian groups.*

Thus a coefficient system $\mathcal{G}$ assigns an abelian group $\mathcal{G}_x$ to each point $x \in X$ and an isomorphism τ_γ from $\mathcal{G}_x$ to $\mathcal{G}_y$ to each path γ from x to y in such a way so that the same isomorphism is attached to homotopic (rel end points) paths.

Example 3.2 *Assume that X is a smooth manifold and E is a flat-bundle over X. Then E induces a local system $\mathcal{E}$ of vector spaces over X. We will abuse notation henceforth and use E to denote both the flat vector bundle and the local system that it induces.*

We now define the homology and cohomology groups of a topological space with coefficients in a local system $\mathcal{G}$ of abelian groups over a topogical space X. We will do this assuming that X is the underlying space of a connected simplicial complex K. We will define the *simplicial* homology and cohomology groups with values in $\mathcal{G}$. By the usual subdivision argument one can prove that the resulting groups are independent of the triangulation K.

We define a p-chain with values in $\mathcal{G}$ to be a formal sum $\Sigma_{i-1}^m \sigma_i \otimes c_i$ where σ_i is an *ordered* p-simplex and c_i is an element of the fiber of $\mathcal{G}$ over the first vertex of σ_i. We denote the group of such chains by $C_p(X, \mathcal{G})$ and define the boundary operator $\partial_p : C_p(X, \mathcal{G}) \longrightarrow C_{p-1}(X, \mathcal{G})$ for $\sigma = (v_0, v_1, \cdots, v_p)$ and c an element of $\mathcal{G}_{v_0}$ by :

$$\partial_p(\sigma \otimes c) = \sigma_0 \otimes \tau_{(v_0, v_1)}(c) + \Sigma_{i=1}^p (-1)^i \sigma_i \otimes c.$$

Here $\tau_{(v_0, v_1)}$ is the isomorphism which is the value of $\mathcal{G}$ on the edge (v_0, v_1) and σ_i, $0 \le i \le p$, is the i-th face of σ. This means that $\sigma_i = (v_0, v_1, \cdots, \hat{v}_i, \cdots, v_p)$ where $\hat{v}_i$ means that the vertex v_i is omitted. Then $\partial_{p-1} \circ \partial_p = 0$ and we define the homology groups $H_*(X, \mathcal{G})$ of X with coefficients in $\mathcal{G}$ in the usual way. These groups depend only on the topological space X and the local coefficient system $\mathcal{G}$.

In a similar way cohomology groups of X with coefficients in $\mathcal{G}$ are defined. A $\mathcal{G}$-valued p-cochain on X with values in $\mathcal{G}$ is a function α which assigns to each ordered p-simplex σ an element $\alpha(\sigma)$ in $\mathcal{G}_{v_0}$ where v_0 is the first vertex of σ. The coboundary $\delta_p \alpha$ of a p-cochain α is defined on a $(p+1)$-cochain σ by :

$$\delta_p \alpha(\sigma) = \tau_{(v_0, v_1)}(\alpha(\sigma_0)) + \Sigma_{i=0}^p (-1)^i \alpha(\sigma_i).$$

Then $\delta_{p+1} \circ \delta_p = 0$ and we define the cohomology groups $H^*(X, \mathcal{G})$ of X with coefficients in $\mathcal{G}$ in the usual way.

In what follows we will need the formulas for ∂ and δ when we express chains and cochains with local coefficients in terms of flat sections of a flat vector bundle E. Note that if t is a flat section of E over a face τ of a simplex σ then it extends to a unique flat section $e_{\sigma,\tau}(t)$ over σ. Similarly if we have a flat section s over σ it restricts to a flat section $r_{\tau,\sigma}(s)$ over τ. Finally if $\sigma = (v_0, \cdots, v_p)$ we define the i-th face σ_i by $\sigma_i = (v_0, \cdots, \hat{v}_i, \cdots, v_p)$. Here $\hat{v}_i$ means the i-th vertex has been omitted.

With these notations we have

$$\partial_p(\sigma \otimes s) = \sum_{i=0}^p (-1)^i \sigma_i \otimes r_{\sigma_i, \sigma}(s)$$

and

$$\delta_p(\alpha(\sigma)) = \sum_{i=0}^{p} (-1)^i e_{\sigma,\sigma_i}(\alpha(\sigma_i)).$$

Remark 3.3 *If we use flat sections as local coefficients we can use* oriented *simplices instead of* ordered *simplices.*

3.1.2 Bilinear pairings

Since our only concern in this paper is with local coefficient systems of vector spaces we will henceforth restrict to that case (although much of the following could be carried out in the above generality). From now on we will regard a simplex with coefficients in $\mathcal{E}$ to be an oriented simplex σ together with a parallel section of $E|\sigma$ (i.e. a flat lift of σ) and a cochain with values in $\mathcal{E}$ to be a rule which assigns to every oriented simplex σ a flat section of the restriction of the bundle E to σ.

Let x_0 be a base–point for X. We first define the Kronecker pairing between homology and cohomology with flat vector bundle coefficients. Let E, F and G be flat bundles over X. Assume that $\nu : E \otimes F \longrightarrow G$ is a parallel section of $Hom(E \otimes F, G)$. Let α be a p-cochain with coefficients in E and $\sigma \otimes c$ be an ordered p-simplex with coeffients in F with $\sigma = (v_0, \cdots, v_p)$. Then the Kronecker index $< \alpha, c >$ is the element of $H_0(X, G)$ defined by:

$$< \alpha, \sigma \otimes c >= \tau_\gamma(\nu(\alpha) \otimes c).$$

Here γ is any path joining v_0 to the base-point x_0. The reader will verify that the right-hand side of the above formula is independent of the choice of γ and that the Kronecker index descends to give a bilinear pairing

$$< \,,\, >: H^p(X, E) \otimes H_p(X, F) \longrightarrow H_0(X, G).$$

We note that if G is trivial then $H_0(X, G) \cong G_{x_0}$. In particular we get a pairing

$$< \,,\, >: H^p(X, E^*) \otimes H_p(X, E) \longrightarrow \mathbb{R}$$

which is easily seen to be perfect (because the chain groups are vector spaces). Thus we have an induced isomorphism

$$H^p(X, E^*) \cong H_p(X, E)^* \text{ or } H_p(X, E^*)^* \cong H^p(X, E).$$

The coefficient pairing $\nu : E \otimes F \to G$ also induces cup products with local coefficients

$$\cup : H^p(X, E) \otimes H^q(X, F) \longrightarrow H^{p+q}(X, G).$$

and cap products with local coefficients(here we assume $m \geq p$)

$$\cap : H^p(X, E) \otimes H_m(X, F) \longrightarrow H_{m-p}(X, G).$$

These are defined in the usual way using the "front-face" and "back-face" of an ordered simplex and pairing the local coefficients using ν.

Finally we will need the following

Theorem 3.4 *Let X be a compact orientable manifold with fundamental class $[X]$. Then we have an isomorphism*

$$\mathcal{D} : H^p(X, E) \longrightarrow H_{n-p}(X, E).$$

given by

$$\mathcal{D}(\sigma) = \sigma \cap [X].$$

Definition 3.5 *Suppose $[a] \in H_p(X, E)$. We will define the* Poincaré dual *of $[a]$ to be denoted $PD([a])$ by*

$$PD([a]) = \mathcal{D}^{-1}([a]).$$

3.2 The de Rham theory of cohomology with local coefficients and the dual of a decomposable cycle

In this subsection we recall the de Rham representations of the cohomology groups $H^*(X, E)$ and of the Poincaré dual class $PD(Y \otimes s)$.

Definition 3.6 *A differential p-form ω with values in a vector bundle E is an section of the bundle $\bigwedge^p T^*(X) \otimes E$ over X.*

Thus ω assigns to a p-tuple of tangent vectors at $x \in X$ a point in the fiber of E over x.

Suppose now that E admits a flat connection ∇.

We can then make the graded vector space of smooth E-differential forms $A^*(X, E)$ into a complex by defining

$$d_\nabla(\omega)(X_1, X_2, \cdots, X_{p+1}) = \sum_{i=1}^{p} (-1)^{i-1} \nabla_{X_i}(\omega(X_1, \cdots, \widehat{X_i}, \cdots, X_{p+1}))$$

$$+ \sum_{i<j} (-1)^{i+j} \omega([X_i, X_j], X_1, \cdots, \widehat{X_i}, \cdots, \widehat{X_j}, \cdots, X_{p+1}). \quad (1)$$

Here $X_i, 1 \le i \le p+1$, is a smooth vector field on X.

The following lemma is standard.

Lemma 3.7

$$d_\nabla^{P+1} \circ d_\nabla^p = 0.$$

We now construct a map ι from $A^p(X, E)$ to the group of simplicial cochains $C^p(X, E)$ as follows. Let $\omega \in A^p(X, E)$ and σ be a p-simplex of K. Then we in a neighborhood U of σ we may write $\omega = \sum_i \omega_i \otimes s_i$ where the s_i's are parallel sections of $E|U$ and the ω_i's are scalar forms. We then define

$$< \iota(\omega), \sigma >= \sum_i (\int_\sigma \omega_i) s_i(v_0).$$

The standard double-complex proof of de Rham's theorem due to Weil, see [BT], page 138, yields

Theorem 3.8 *The integration map $\iota : H^*_{deRham}(X, E) \longrightarrow H^*(X, E)$ is an isomorphism.*

We will need a special representation in de Rham cohomology with coefficients in E of the cohomology class $PD(Y \otimes s)$. Here we assume that Y is a compact oriented p-dimensional submanifold of X and s is a nonzero parallel section of $E|Y$.

Suppose then that the Poincaré dual cohomology class of the homology class carried by Y is represented in de Rham cohomology of X by the closed $n-p$-form ω_Y where ω_Y is supported in a tubular neighborhood U of Y. The parallel section s of $E|Y$ extends to a parallel section of $E|U$ again denoted s. We extend $\omega_Y \otimes s$ to M by making it zero outside of U. We continue to use the notation $\omega_Y \otimes s$ for this extended form. We note that it is standard that the extension of ω_Y by zero represents the Poincaré dual to $[Y]$.

Lemma 3.9 *Then the de Rham cohomology class Poincaré dual to the cycle with coefficients $Y \otimes s$ is represented by the bundle–valued form $\omega_Y \otimes s$.*

Proof. We are required to prove that for any E^*-valued closed p–form η we have

$$\int_M \eta \wedge \omega_Y \otimes s = \int_{Y \otimes s} \eta = \int_Y \langle \eta, s \rangle.$$

But

$$\int_M \eta \wedge \omega_Y \otimes s = \int_M \langle \eta, s \rangle \wedge \omega.$$

But because s is parallel on U the scalar form $\langle \eta, s \rangle$ is closed and the lemma follows because ω_Y is the Poincaré dual to $[Y]$. $\square$

3.2.1 The intersection theory of cycles with local coefficients

From now on we assume that X is a compact oriented n–manifold and E, F and G are flat vector bundles over X and $\nu : E \otimes F \longrightarrow G$ is a parallel section of $Hom(E \otimes F, G)$. We orient the top-dimensional simplices of X so that their boundaries cancel. Let $[a] \in H_p(X, E)$ and $[b] \in H_q(X, F)$.

Definition 3.10 *The intersection product $[a] \cdot [b] \in H_{p+q-n}(X, G)$ of $[a]$ and $[b]$ is defined by*
$$[a] \cdot [b] = \mathcal{D}(PD([b]) \cup PD([a])),$$

Our goal for the rest of this section is to give a geometric meaning to the intersection product. By this we mean we would like to be able to represent it by a simplicial cycle supported on the intersection of the simplices comprising a and b where a and b are appropriate representatives of the classes $[a]$ and $[b]$ and we want this intersection to have an appropriate structure (i.e. a subcomplex of the correct dimension). First we need to discuss what it means for representing simplicial cycles a for $[a]$ and b for $[b]$ to be in *general position*. In what follows if c is a simplicial chain we define the carrier $|c|$ to be the subset of X which is the union of the simplices of c.

Definition 3.11 *We will say that a p-simplex a is in general position to a q-simplex b if the intersection $a \cap b$ is (a simplex) of dimension less than or equal to $p + q - n$.*

We say the simplicial cycles $a = \sum_i a_i$ (of dimension p) and $b = \sum_j b_j$ (of dimension q) are in general position if, for all i, j, the simplices a_i and b_j are in general position. Hence the intersections $a_i \cap b_j$ satisfy

$$dim(a_i \cap b_j) \leq p + q - n.$$

Given two simplicial cycles a and b we can move $|a|$ by an ambient isotopy, one simplex at a time, so that each moved simplex, $\bar{a}_i$, is in general position to each simplex b_j of b by [Hu], Lemma 4.6. Let $|\bar{a}|$ be the union of the moved simplices (the moved simplices are not necessarily simplices in X). Thus we have $h_t : |a| \to X$ with h_0 equal to the inclusion of $|a|$ in X and the image of h_1 equal to $|\bar{a}|$. The proof of [Hu], Lemma 4.6 shows that we may choose h_1 to be a piecewise linear embedding. We may regard the pair a and h_1 as a singular cycle homologous to the original simplicial cycle a in the singular complex.

Finally, according to [Hu], Lemma 1.10, we may subdivide a and X so that h_1 is simplicial. We use $\tilde{a}$ to denote the subdivided image of h_1. It is automatic , [Hu], Lemma 1.3, that the resulting subdivision of b is also a subcomplex. Hence the intersection $|\tilde{a}| \cap |b|$ is also a subcomplex. We will label the simplices in $\tilde{a}$ as $\tilde{a}_i$ and label the simplices in the subdivision of b by b_j so i and j run through new index sets. Thus we have arranged that, for all i, j, $\tilde{a}_i$ and b_j are in general position and that $|\tilde{a}|, |b|$ and the intersection $|\tilde{a}| \cap |b|$ are subcomplexes. It follows that the intersection of $|\tilde{a}|$ and $|b|$ is a union of simplices $\{c_{ij} := \tilde{a}_i \cap b_j\}$ such that for each pair i, j, c_{ij} is a face common to $\tilde{a}_i$ and b_j. We use the simplicial map h_1 to transfer the orientation of each simplex of the subdivision of a to its image in $\tilde{a}$ to obtain a simplicial cycle. Since h_1 is homotopic to the inclusion of a this cycle is homologous to a.

We wish to assign an orientation to each simplex c_{ij} in the intersection. If $dim(c_{ij}) < p + q - n$ we assign the coefficient zero to it. Otherwise we first choose an orientation of c_{ij}. Since $\tilde{a}_i$ is oriented and c_{ij} is oriented we obtain an induced orientation on $Lk(c_{ij}, \tilde{a}_i)$, the link of c_{ij} in $\tilde{a}_i$ and on $Lk(c_{ij}, b_j)$. Now the star $St(c_{ij})$ is an n-ball and has an orientation induced by the orientation of X. The union of the vertices of $c_{ij}, Lk(c_{ij}, \tilde{a}_i)$ and $Lk(c_{ij}, b_j)$ spans a subpolyhedron P of $St(c_{ij})$ with nonempty interior . We arrange the vertices so that each of the three subsets of vertices induces the correct orientation on c_{ij}, resp. $Lk(c_{ij}, \tilde{a}_i)$, resp. $Lk(c_{ij}, b_j)$. We orient P by taking the vertices in the order given and assign the coefficient $+1$ to c_{ij} with the chosen orientation if this orientation agrees with the orientation P receives from the orientation of $St(c_{ij})$. Otherwise we assign the coefficient -1 (or equivalently we take the opposite orientation on c_{ij}).

We can now give a geometric definition of the intersection product of simplicial cycles with local coefficients.

1. Subdivide and choose representing simplicial chains $a = \sum_i a_i \otimes s_i$ and $b = \sum_j b_j \otimes t_j$ in general position.

2. Assign to each simplex c_{ij} of the intersection an orientation according to the rule above.

3. Give each c_{ij} as coefficient the parallel section of G over c_{ij} obtained from the parallel sections s_i and t_j by restricting each of them to c_{ij} and applying ν.

We will (temporarily) denote the intersection product of $[a]$ and $[b]$ defined in this way by $[a] \circ [b]$.

Theorem 3.12 *The geometric definition of the intersection product agrees with the cup-product definition.*

$$[a] \cdot [b] = [a] \circ [b].$$

We will not prove this theorem but will prove the special case of it we need, see Theorem 3.15.

3.3   Decomposable cycles and an intersection formula

There is a particularly simple construction of cycles with coefficients in E. Let Y be a closed, oriented submanifold of X of codimension p and let s be a parallel section of the restriction of E to Y. Let $[Y]$ denote the fundamental cycle of Y so $[Y] = \Sigma_i \sigma_i$, a sum of ordered simplices.

Definition 3.13 $Y \otimes s$ *denotes the p-chain with values in E given by*

$$Y = \Sigma_i \sigma_i \otimes s_i$$

where s_i is the value of s on the first vertex of σ_i.

Lemma 3.14 $Y \otimes s$ *is an p cycle.*

Proof. Let τ be an $p-1$ simplex in Y. Then the coefficient of τ in the boundary of $Y \otimes s$ is the product of the coefficient of τ in the boundary of the fundamental cycle times the value of s on the first vertex of τ. But the first factor of the this product is zero. □

We will refer to such cycles henceforth as *decomposable* cycles. The terminology is motivated by the case of 1-cycles and their connection with the Eilenberg-MacLane complex as we will explain in the next subsection.

We now specialize the intersection product of cycles with local coefficients to cycles of the above type. Suppose that Y_1 and Y_2 are closed, oriented submanifolds of dimensions p and q respectively, s_1 and s_2 are parallel sections of $E|Y_1$ and $F|Y_2$ respectively and Y_1 and Y_2 intersect transversally. Then we may simplify the previous formula defining the intersection class $(Y_1 \otimes s_1) \cdot (Y_2 \otimes s_2)$ as follows

1. Intersect Y_1 and Y_2 in the usual way to obtain a (possibly disconnected) oriented dimension $p + q - n$ manifold $Z = \coprod_i Z_i$ with intersection multiplicity $\epsilon_i = \pm 1$ along Z_i.

2. Assign to Z_i the parallel section of $G|Z_i$ given by $\nu(s_1|Z_i, s_2|Z_i)$.

Then

Theorem 3.15

$$(Y_1 \otimes s_1) \cdot (Y_2 \otimes s_2) = \sum_i \epsilon_i Z_i \otimes \nu(s_1|Z_i, s_2|Z_i)$$

Proof.

By Lemma 3.9 we have

$$PD(Y_1 \otimes s_1) = \omega_{Y_1} \otimes s_1$$

and

$$PD(Y_2 \otimes s_2) = \omega_{Y_2} \otimes s_2.$$

Now by definition we have

$$PD(Y_1 \otimes s_1 \cdot Y_2 \otimes s_2) = \omega_{Y_2} \otimes s_2 \wedge \omega_{Y_1} \otimes s_1 = \omega_{Y_2} \wedge \omega_{Y_1} \otimes \nu(s_1, s_2).$$

But it is well-known, see [Br], Theorem 11.7, that $PD(\sum_i \epsilon_i Z_i) = \omega_{Y_2} \wedge \omega_{Y_1}$, We again apply Lemma 3.9 to obtain

$$PD(\sum_i \epsilon_i Z_i \otimes \nu(s_1|Z_i, s_2|Z_i)) = \omega_{Y_2} \wedge \omega_{Y_1} \otimes \nu(s_1, s_2).$$

$\square$

3.3.1 Decomposable cycles and the Eilenberg-MacLane complex

In this subsection we will assume that manifolds X and Y are Eilenberg-MacLane spaces and will show that decomposable cycles $Y \otimes s$ have representatives of a particularly simple type in the Eilenberg-MacLane complex $C_.(\pi_1(X)) \otimes V$. Let Γ be the fundamental group of X. The flat bundle E corresponds to a Γ-module V. The one dimensional simplicial homology group $H_1(X, E)$ defined above) is isomorphic to the Eilenberg-Maclane (i.e.group) homology group $H_1(\Gamma, V)$. Let $\mathbb{R}(\Gamma)$ be the real group ring of Γ. A one–chain in the Eilenberg-MacLane complex of Γ with values in V is an element of the tensor product $\mathbb{R}(\Gamma) \otimes V$; that is, a sum $c = \sum_i \gamma_i \otimes v_i$. The formula for the boundary of the above Eilenberg-MacLane chain is

$$\partial_1(c) = \sum_i \gamma_i \cdot v_i - v_i.$$

Thus the simplest kind of Eilenberg-MacLane one-cycle is a decomposable element $\gamma \otimes v$ where $\gamma \cdot v = v$. Such cycles (decomposable elements of the group of one chains) were called decomposable cycles in [GM] (see also [KaM] for the connection with Eichler-Shimura periods of classical modular forms). Now let c be a smooth curve representing γ. In case c can be chosen to be embedded then the vector invariant vector v corresponds to a parallel section

s along c and $\gamma \otimes v$ corresponds under the above isomorphism to the cycle $c \otimes s$ above. Of course from the point of view of simplical chains such cycles are not decomposable elements but they are "as decomposable as possible" whence our terminology.

Assume now that X is a closed oriented n-manifold which is an Eilenberg-MacLane space of type $K(\Gamma, 1)$ and Y is a closed oriented m-dimensional submanifold which is an Eilenberg-MacLane space of type $K(\Phi, 1)$ (this will be the case for the cycles studied in this paper). We will now see that the m-cycle $Y \otimes s$ again has a nice interpretation in the Eilenberg-MacLane chain complex $C_.(\Gamma) \otimes V$. Indeed, see [McL], page 114, (5.3), we recall that the Eilenberg-MacLane boundary operator $\partial : C_p(\Gamma) \otimes V \to C_{p-1}(\Gamma) \otimes V$ is given by the formula

$$\partial((\gamma_1, \gamma_2, \cdots, \gamma_p) \otimes v) = (\gamma_2, \cdots, \gamma_p) \otimes \gamma_1 \cdot v +$$
$$\sum_{i=1}^{p-1} (-1)^i (\gamma_1, \cdots, \gamma_i \gamma_{i+1}, \cdots, \cdots, \gamma_p) \otimes v + (-1)^p (\gamma_1, \cdots, \gamma_{p-1}) \otimes v.$$

From this we see that if $[\Phi]$ is the fundamental cycle of Y expressed in the Eilenberg-MacLane complex $C_m(\Phi)$ and v is the Φ-invariant vector in V corresponding to the section s then in the complex $C_m(\Phi) \otimes V$ we have

$$\partial([\Phi] \otimes v) = (\partial[\Phi]) \otimes v = 0.$$

Let $f : \Phi \to \Gamma$ be the map on fundamental groups induced by the inclusion $Y \subset X$. Then f induces chain maps $\widehat{f} : C_.(\Phi) \to C_.(\Gamma)$ and $\widehat{f} \otimes 1 : C_.(\Phi) \otimes V \to C_.(\Gamma) \otimes V$. Indeed, if σ is an Eilenberg-MacLane p-simplex for Φ, we let $\widehat{f}(\sigma)$ denote the simplex (for Γ) obtained by replacing each group element in σ with its image under f. We extend $\widehat{f}$ linearly to chains. Then the m-cycle with local coefficients $Y \otimes s$ corresponds to the Eilenberg-MacLane m-cycle $\widehat{f}[\Phi] \otimes v$.

4 The standard arithmetic quotients of $\mathbb{H}^n$ and their decomposable cycles with coefficients

4.1 The standard arithmetic quotients $M = \Gamma \backslash \mathbb{H}^n$

let $\mathbb{K}$ be a totally-real number field with archimedean completions $\{v_0, \cdots, v_r\}$. Let $\mathcal{O}$ be the ring of algebraic integers in $\mathbb{K}$. Let $\underline{V}$ be an oriented vector space over $\mathbb{K}$ of dimension $n + 1$. Let V be the completion of $\underline{V}$ at v_0. Let $q : V \to \mathbb{K}$ be a quadratic form such that q has signature $(n, 1)$ at the completion v_0 and is positive-definite at all other completions. In fact we will restrict ourselves to the example of the Introduction, that is:

$$f(x_1, \cdots, x_{n+1}) = x_1^2 + x_2^2 + \cdots + x_n^2 - \sqrt{m} x_{n+1}^2.$$

Let $\underline{G}$ be the algebraic group of whose $\mathbb{K}$-points is the group of orientation preserving isometries of f. Let L be an $\mathcal{O}$ lattice in V and let $\Phi = \underline{G}(\mathcal{O})$ be the subgroup of $\underline{G}(\mathbb{K})$ consisting of those elements that take L into itself. We define $G := \underline{G}(\mathbb{R})$. We choose a base-point o in $\mathbb{H}^n$ and let K be the maximal compact subgroup of G that fixes o. We let $\mathfrak{b}$ be an ideal in $\mathcal{O}$ and let $\Gamma = \Gamma(\mathfrak{b})$ be the congruence subgroup of Φ of level $\mathfrak{b}$ (that is the elements of Φ that are congruent to the identity modulo $\mathfrak{b}$).

In what follows it will be convenient to take the projective model for $\mathbb{H}^n$. We let $D \subset \mathbb{P}(V)$ be the set of "negative lines", that is

$$D = \{z \in \mathbb{P}(V) : (\ ,\ )|z \le 0\}.$$

We define $z_0 = e_{n+1}$ and

$$V_+ = (\mathbb{R}z_0)^{\perp}.$$

We take z_0 to be the base-point o.

4.2   Totally-geodesic submanifolds in the standard arithmetic examples

Let $x = \{x_1, x_2, \cdots, x_k\} \in \underline{V}^k$ (i.e. a k-tuple of $\mathbb{K}$ –rational vectors). We assume that $X := span\ x$ has dimension k and moreover $(\ ,\ )|X$ is positive definite. In what follows we will let r_X be the isometric involution of V given by

$$r_X(v) = \begin{cases} -v & v \in X \\ v & v \in X^{\perp}. \end{cases}$$

We define the totally-geodesic subsymmetric space D_X by

$$D_X = \{\mathbb{R}z \in D : (z, x_i) = 0, 1 \le i \le p\}.$$

Then D_X is the fixed-point set of r_X acting on D.

We also define subgroups G_X (resp. Γ_X) to be the stabilizer in G (resp. in Γ) of the subspace X. We define $G'_X \subset G_X$ to be the subgroup that acts trivially on X.

We then have, see [JM], Lemma 7.1 or [FOR],Lemma 2.4.

Lemma 4.1 *There exists a congruence subgroup* $\Gamma := \Gamma(\mathfrak{b})$ *of* Φ *such that*

1. *$M := \Gamma\backslash D$ is an orientable compact manifold of codimension k.*

2. *The image M_X of D_X in M is the quotient $\Gamma_X\backslash D_X$ and is an orientable submanifold.*

In what follows we will require $x \in V_+^k$. Accordingly we have $z_0 \in D_X$.

4.3 Attaching a local coefficient s_x to the cycle M_X

4.3.1 The existence of a parallel section

We now want to promote M_X to a (decomposable) cycle with coefficients for appropriate coefficient systems W by finding a nonzero parallel section of s_X of $\widetilde{W}|M_X$. The main point is

Lemma 4.2 *If Γ is a neat subgroup ([B],pg.117) then Γ_X acts trivially on X.*

Proof. We have a projection map $p_X : \Gamma_X \to O(X_1) \times O(X_2) \times \cdots \times O(X_r)$. Here by X_i we mean the i-th completion of X. The i-th completion of $(\ ,\ )$ restricted to X_i is positive definite for $1 \leq i \leq r$. Furthermore the splitting $V = X \oplus X^\perp$ is defined over $\mathbb{K}$. Thus the diagonal embedding of the intersection $L_X = L \cap X$ is a lattice in $\oplus_{i=0}^{r} X_i$ which is invariant under $p_X(\Gamma_X)$. Hence $p_X(\Gamma_X)$ is a discrete subgroup of a compact group hence a finite group. Hence if $\gamma \in p_X(\Gamma_X)$ then all eigenvalues of γ are roots of unity. Since Γ is neat all eigenvalues must be 1 and the lemma follows. $\square$

We can now prove the existence of a nonzero parallel section along M_X provided that $dim(X)$ is big enough. For economy of notation we will adopt the following terminology up to the end of the following proposition. We will say a dominant weight μ for $SO(2m-1,1)$ is *admissible* if $i(\mu) \leq m-1$. For the case $SO(2m,1)$ all dominant weights will be defined to be admissible. We note by Theorem 2.1 that if the highest weight of W is not admissible then all cohomology groups with coefficients in W vanish.

Proposition 4.3 *Suppose that Γ is neat and μ is admissible. Then there exists a nonzero vector $\tau \in S_{[\mu]}V$ invariant under $\Gamma_X \Leftrightarrow dim(X) \geq i(\mu)$.*

Proof. We apply the Gelfand-Tsetlin branching theorem for the pair of groups $SO(m-1) \subset SO(m)$, [Bo], pg. 267 and pg. 269 to find that if the irreducible representation of $SO(m-1)$ with highest weight ν occurs in the restriction to $SO(m-1)$ of the representation of $SO(m)$ with admissible highest weight μ then

$$i(\mu) - 1 \leq i(\nu) \leq i(\mu) \text{ and } i(\nu) = i(\mu) - 1 \text{ is realized.}$$

Thus we have to branch at least $i(\mu)$ times to get the zero weight and if we branch $i(\mu)$ times then the zero weight occurs. $\square$

In what follows we will need an explicit formula for an invariant vector because we will have to take the inner product of two such invariants. The rest of this section is devoted to finding such an explicit formula. However the result that in order to find a parallel section it is necessary that

$$dim(X) \geq i(\mu)$$

will play a critical role throughout the rest of the paper.

We will use Weyl's construction of the irreducible representations of $O(V)$ to find an explicit formula for a parallel section. We first review Weyl's construction.

4.3.2   The harmonic Schur functors

We will follow [FH] in our description of the harmonic Schur functor $U \to S_{[\mu]}U$ corresponding to a partition μ on quadratic spaces $U, (,)$. Here $(,)$ is assumed to be nondegenerate. There is also a nice treatment in [GW]. We assume that U has dimension m.

Suppose then that W_μ is the irreducible representation of $SO(U)$ with highest weight $\mu = (b_1, b_2, \cdots, b_l)$ where $l = [\frac{m}{2}]$. We will abuse notation and use μ to denote the corresponding partition of $d = \sum b_i$. We extend the quadratic form $(,)$ to $\otimes^d U$ as the d-fold tensor product and note that the action of S_d on $\otimes^d U$ is by isometries. For each pair $I = (i, j)$ of integers between 1 and d we have the contraction operator $\Phi_I : \otimes^d U \to \otimes^{d-2}U$, the operator $\Psi_I : \otimes^{d-2}U \to \otimes^d U$ that inserts the (dual of the form) form $(,)$ an into the (i, j)-th spots and the composition $\theta_I := \Psi_i \circ \Phi_I$, (the notation is that of [FH]). We define the harmonic d-tensors, to be denoted $U^{[d]}$, to be the kernel of all the contractions Φ_I. Following [FH], pg. 263, we define the subspace $U^{[d]}_{d-2r}$ of $U^{\otimes d}$ by

$$U^{[d]}_{d-2r} = \sum \Psi_{I_1} \circ \cdots \Psi_{I_r} U^{[d-2r]}.$$

Carrying over the proof of [FH], Lemma 17.15 (and the exercise that follows it) from the symplectic case to the orthogonal case we have

Lemma 4.4 *We have a direct sum, orthogonal for $(,)$,*

$$U^{\otimes d} = U^{[d]} \oplus \oplus_{r=1}^{[\frac{d}{2}]} U^{[d]}_{d-2r}.$$

We define the harmonic projection $\mathcal{H} : \otimes V \to \otimes V$ to be the orthogonal projection on the harmonic d-tensors $U^{[d]}$. The space of harmonic d-tensors $U^{[d]}$ is invariant under the action of S_d. Consequently we may apply the idempotents in the group algebra of S_d corresponding to partitions to further decompose $U^{[d]}$ as an $SO(U)$–module.

We use μ to denote the associated partition of d. *We note that $i(\mu)$ equals the length of the first column of μ.* We label the Young diagram corresponding to the partition μ in the standard way, i.e. by putting $1, 2, \cdots, b_1$ in the first row and continuing in the obvious way to obtain a standard tableau T. Let P (resp. Q) be the group preserving the rows (resp. columns) of T. Define elements of the group ring of S_d by $\mathcal{P} = \sum_P p$ and $\mathcal{Q} = \sum_Q \epsilon(q)q$. Again following [FH], pg. 296, we define the harmonic Schur functor $S_{[\mu]}U$ as follows.

Definition 4.5
$$S_{[\mu]}U = \mathcal{QP}U^{[d]}.$$

We then have the following theorem, [FH], Theorem 19.19

Theorem 4.6 *The space $S_{[\mu]}U$ is an irreducible representation of $O(V)$. It is nonzero if and only if the sum of the first two columns of the partition μ is less than or equal to $dim(U)$.*

In what follows we will need the following

Lemma 4.7   *1. $\mathcal{H}, \mathcal{P}$ and $\mathcal{Q}$ are self-adjoint relative to $(\ ,\ )$.*

2. $\mathcal{H}$ commutes with $\mathcal{P}$ and $\mathcal{Q}$.

Proof. It is clear that $\mathcal{H}$ is self-adjoint. The arguments for $\mathcal{P}$ and $\mathcal{Q}$ are the same. We give the one for $\mathcal{Q}$. We will use the symbol q to denote both the element $q \in Q$ and the corresponding operator on $V^{\otimes d}$. Since q is an isometry we have $q^* = q^{-1}$. Hence we have

$$\mathcal{Q}^* = \sum \epsilon(q) q^* = \sum \epsilon(q^{-1}) q^{-1} = \mathcal{Q}.$$

To prove that $\mathcal{H}$ commutes with $\mathcal{P}$ and $\mathcal{Q}$ it suffices to prove that $\mathcal{H}$ commutes with every element $g \in S_d$. But S_d acts by isometries and preserves $U^{[d]}$. Consequently it commutes with orthogonal projection on $U^{[d]}$. $\qquad\square$

4.3.3   An explicit formula for a parallel section

We will now return to the case in hand. Assume that $i(\mu) \le k \le m = [\frac{n+1}{2}]$. We now construct an explicit parallel section s_e where $e = (e_1, e_2, \cdots, e_k)$. Suppose that $b_i = 0, i \ge k+1$. Put $E_k = span\ \{e_1, \cdots, e_k\}$. Define an element $\sigma_e \in S^{b_1} E_k \otimes S^{b_2} E_k \otimes \cdots S^{b_k} E_k$ by

$$\sigma_e = e_1^{\otimes b_1} \otimes \cdots e_k^{\otimes b_k}.$$

Clearly we have $\mathcal{P}\sigma_e = \sigma_e$.
   We define $\tau_e \in S_{[\mu]} V$ by

$$\tau_e = \mathcal{Q}\mathcal{P}\mathcal{H}\sigma_e.$$

We observe that $\mathcal{H}\sigma_e \notin \otimes E_k$ and $\tau_e \notin S_{[\mu]} E_k$ because the harmonic projection $\mathcal{H}$ corresponding to the pair $V, (\ ,\ )$ does not carry $S^*(E_k)$ into itself. We have

Lemma 4.8 τ_e *is* G'_{E_k}*–invariant.*

Proof. $g \cdot \mathcal{Q}\mathcal{P}\mathcal{H}\sigma_e = \mathcal{Q}\mathcal{P}\mathcal{H}g \cdot \sigma_e = \mathcal{Q}\mathcal{P}\mathcal{H}\sigma_e.$ $\qquad\square$

Continue to assume that $i(\mu) \le k \le m = [\frac{n+1}{2}]$. Recall that we previously defined elements $u_i, v_i \in V, 1 \le i \le m = [\frac{n+1}{2}]$ by $u_i = e_i - \imath e_{m+i}, v_i = e_i + \imath e_{m+i}, 1 \le i \le m-1$. Furthermore we have

$$u_m = \begin{cases} e_m - \imath e_{2m}, n = 2m \\ e_m - e_{2m}, n = 2m-1 \end{cases}$$

and

$$v_m = \begin{cases} e_m + \imath e_{2m}, n = 2m \\ e_m + e_{2m}, n = 2m-1. \end{cases}$$

We next define the element $\sigma_u \in S_{[\mu]} V \otimes \mathbb{C}$ by

$$\sigma_u = u_1^{\otimes b_1} \otimes \cdots u_k^{\otimes b_k}.$$

We recall that $(u_i, u_j) = 0, 1 \le i, j \le k$ and we obtain

Lemma 4.9 σ_u *is a harmonic tensor.*

Corollary 4.10 $\tau_u = \mathcal{Q}\sigma_u = \mathcal{QPH}\sigma_u \in S_{[\mu]}X \otimes \mathbb{C}$.

Remark 4.11 *The vector τ_u is a weight vector of weight $(b_1, \cdots, b_m)$ and consequently is a highest weight vector for W.*

We can now prove that τ_e and τ_u are both nonzero.

Lemma 4.12 $(\tau_e, \tau_u) = 1$.

Proof.

$$(\tau_e, \tau_u) = (\mathcal{QPH}\sigma_e, \mathcal{Q}\sigma_u) = (\mathcal{PH}\sigma_e, \mathcal{Q}\sigma_u) = (\mathcal{HP}\sigma_e, \mathcal{Q}\sigma_u) = (\sigma_e, \mathcal{HQ}\sigma_u) = (\sigma_e, \mathcal{Q}\sigma_u).$$

Next we note that $(e_i, u_j) = \delta_{ij}$. Hence for $q \in Q_\mu$ we have

$$(\sigma_e, q\sigma_u) = \left\{ \begin{array}{ll} 1, & q = 1 \\ 0, & q \neq 1. \end{array} \right.$$

$\square$

We can now give a formula for a nonzero invariant. We have proved

Theorem 4.13 *Suppose that $X \subset V_+$ satisfies $i(\mu) \leq dim(X) = k \leq m = [\frac{n+1}{2}]$. Assume that $\mu = (b_1, \cdots, b_k, 0, \cdots, 0)$. Suppose that $x = (x_1, \cdots, x_k)$ is an orthogonal basis for X. Define*

$$\tau_x = \mathcal{QPH}(x_1^{b_i} \otimes \cdots \otimes x_i^{b_k}).$$

Then τ_x is an nonzero G'_X–invariant in $S_{[\mu]}V$.

Finally we will need to find k-tuples $y = (y_1, \cdots, y_k) \in V^k$ such that if we define $\sigma_y = y_1^{\otimes b_1} \otimes \cdots y_k^{\otimes b_k}$ and $\tau_y = \mathcal{QPH}\sigma_y$ we have

Lemma 4.14
$$(\tau_x, \tau_y) \neq 0.$$

Proof. There exist $s_1 \in S^{b_1}V, \cdots, s_k \in S^{b_k}V$ such that $(\tau_x, \mathcal{QPH}(s_1 \otimes \cdots s_k)) \neq 0$. But the pure powers $v^m, v \in V$ span S^mV. Hence, we conclude that there exist $y_1, \cdots y_k \in V$ such that

$$(\tau_x, \mathcal{QPH}(y_1^{\otimes b_1} \otimes \cdots \otimes y_k^{\otimes b_k})) \neq 0.$$

$\square$

5 Configurations of compact totally-geodesic submanifolds in the standard arithmetic examples

In this section we will review the results of [JM] and [FOR].

Let $x = \{x_1, x_2, \cdots, x_k\}$ and $y = \{y_1, y_2, \cdots, y_l\}$ be such that $z = x \cup y$ spans a subspace Z of V of dimension $k + l$ such that the restriction of $(\ ,\ )$ to Z is positive definite. In what follows we let $X = span\ x$ and $Y = span\ y$.

We have the Theorem 7.2 of [JM], see also Corollary 2.26 of [FOR].

Theorem 5.1 *There exists a congruence cover M' of M such that $M'_X \cap M'_Y$ consists of the single component $M'_Z := \pi'(D_Z)$. Moreover for any congruence cover M'' of M' the intersection $M''_X \cap M''_Y$ again consists of the single component $M''_Z := \pi''(D_Z)$*

Corollary 5.2 *Suppose $k + l = n$. Then there exists a congruence cover M' of M such that $M'_X \cap M'_Y$ consists of the single point $\pi'(z_0)$.*

6 The nonvanishing theorem

We can now prove the desired nonvanishing theorem. Suppose W is an irreducible representation of $SO(n, 1)$ with highest weight $\mu = (b_1, \cdots, b_m)$. Here we assume that if n is even then $n = 2m$ and if n is odd then $n = 2m-1$. Assume $i(\mu) \leq k \leq n$ and choose $x = e = (e_1, \cdots, e_k)$. We have shown that there exists a nonzero parallel section s_e of $\widetilde{W}|M_X$ corresponding to the Γ_X-invariant vector $\tau_e \in W$. But we will also need a parallel section over a complementary cycle M_Y. Hence by Proposition 4.3 we require

$$k \geq i(\mu) \text{ and } n - k \geq i(\mu).$$

Hence we require $i(\mu) \leq [\frac{n}{2}]$. We will also need to construct the section s_y from a k-element subset of a spanning set y for Y. Accordingly we will assume henceforth that $k \leq n - k$ or $k \leq [\frac{n}{2}]$.

Lemma 6.1 *Assume now that $i(\mu) \leq k \leq [\frac{n}{2}]$ whence $k \leq n-k$. Then there exist (infinitely many) $\mathbb{K}$-rational $n - k$-tuples $y = (y_1, \cdots, y_{n-k}) \in V^{n-k}$ such that if we define $y' = (y_1, \cdots, y_k)$ then we have*

1. *$(\tau_e, \tau_{y'}) \neq 0$*

2. *Let $z = \{x_1, \cdots, x_k, y_1, \cdots, y_{n-k}\}$ and $Z = span\ z$. Then $dim(Z) = n$.*

3. *The restriction of $(\ ,\ )$ to Z is positive definite.*

Proof. We claim that the negation of each of the first two conditions defines a *proper* algebraic subvariety of V^{n-k}. For the second the claim is obvious. The first requires some more work.

Define $\Phi : V^{n-k} \to S_{[\mu]}V$ by

$$\Phi(y) = \tau_{y'} := \mathcal{QPH}(y_1^{\otimes b_i} \otimes \cdots \otimes y_k^{\otimes b_k}).$$

Then Φ is a polynomial mapping. We define a polynomial P_e on V^{n-k} by

$$P_e(y) = (\tau_e, \tau_{y'}).$$

But in Lemma 4.14 we have proved that there exists $y \in V^{n-k}$ such that $P_e((y_1, \cdots, y_{n-k})) = (\tau_e, \tau_{y'}) \neq 0$ Thus P_e is not identically zero and the zero set of P_e (the negation of 1.) is a proper algebraic hypersurface.

Thus the set of y's satisfying each of the first two conditions is nonempty, open and dense. Finally the set of $y's$ satisfying the third condition is open (in the classical topology) and consequently the set of points satisfying all three conditions is a nonempty open subset of V^{n-k}. Hence, it contains infinitely many rational points.

$\square$

We can now prove a preliminary nonvanishing result.

Lemma 6.2 *Assume $i(\mu) \leq k \leq [\frac{n}{2}]$. Then $H^k(\Gamma, W) \neq 0$.*

Proof. Let $X = E_k$ with $k \leq [\frac{n}{2}]$. Choose $\{y_1, \cdots, y_{n-k}\} \subset V^{n-k}$ satisfying the three conditions of Lemma 6.1 of the previous lemma. Now apply Theorem 5.1 and choose a neat congruence subgroup of Γ so that $M_X \cap M_Y$ consists of single point. Then

$$M_X \otimes s_x \cdot M_Y \otimes s_y = (s_x, s_y) = (\tau_e, \tau_y) \neq 0.$$

Since W is self-dual we have $H_k(\Gamma, W)^* = H^k(\Gamma, W) \neq 0$. $\square$

Remark 6.3 *We see that the method fails (as it should) for the case of weights μ with $i(\mu) = m$ for $SO(2m - 1, 1)$. Indeed in this case we need both the cardinalities of x and y to be at least m (in order that the coefficients τ_x and τ_y in $S_{[\mu]}V$ exist). But $(\ ,\ )|(X + Y)$ positive definite implies $\dim(X + Y) \leq 2m - 1$ and consequently $X \cap Y \neq 0$.*

Now we have the required nonvanishing theorem.

Theorem 6.4   *1. Suppose that n is even. Then*

$$H^k(\Gamma, W) \neq 0, \ i(\mu) \leq k \leq n - i(\mu).$$

2. Suppose n is odd and $i(\mu) < \frac{n+1}{2}$. Then

$$H^k(\Gamma, W) \neq 0, \ i(\mu) \leq k \leq n - i(\mu).$$

Proof. By Poincaré duality

$$H^k(\Gamma, W) \cong H^{n-k}(\Gamma, W).$$

But by the previous lemma we have shown that we have nonvanishing up to and including the middle dimension in case n is even and up to and including the first of the two middle dimensions in case n is odd. $\square$

7 Nonvanishing of cup-products

We have

Theorem 7.1 *Let W_1 and W_2 be irreducible representations with highest weights μ_1 and μ_2 satisfying $i(\mu_1) = p_1$ and $i(\mu_2) = p_2$. Suppose that $q_1 \geq p_1$ and $q_2 \geq p_2$ and $q_1 + q_2 \leq [\frac{n}{2}]$. Then (for $\mathfrak{b}$ depending on W_1 and W_2) the cup-product*

$$H^{q_1}(\Gamma, W_1) \otimes H^{q_2}(\Gamma, W_1) \to H^{q_1+q_2}(\Gamma, W_1 \otimes W_2)$$

is a nonzero map. Furthermore if $\pi : W_1 \otimes W_2 \to W$ is a homomorphism of $SO(n,1)$-modules such that $\pi(\tau_{x'} \otimes \tau_{x'}) \neq 0$ then the induced cup-product with values in W is also nonzero.

The proof will follow the same lines as the proof of Theorem 6.4.

Choose $x = (e_1, \cdots, e_{q_1})$ and $y = (e_{q_1+1}, \cdots, e_{q_1+q_2})$. We put $x' = (e_1, \cdots, e_{p_1})$ and $y' = (e_{q_1+1}, \cdots, e_{q_1+p_2})$ and $z = (e_1, \cdots, e_{q_1}, e_{q_1+1}, \cdots, e_{q_1+q_2})$. We define $\tau_{x'} \in W_1$ and $\tau_{y'} \in W_2$ as before. We obtain corresponding parallel sections $s_{x'}$ and $s_{y'}$ of $\widetilde{W_1}|M_X$ (resp. $\widetilde{W_2}|M_Y$).

We also have the totally geodesic submanifold M_Z where $Z = span\ z$. After passing to a sufficiently deep congruence subgroup we have by Theorem 5.1

$$M_X \otimes s_{x'} \cdot M_Y \otimes s_{y'} = M_Z \otimes (s_{x'} \otimes s_{y'}).$$

Here we have used the natural map ϕ from $\Gamma(M_Z, \widetilde{W_1}|M_Z) \otimes \Gamma(M_Z, \widetilde{W_2}|M_Z)$ into $\Gamma(M_Z, \widetilde{W_1}|M_Z \otimes \widetilde{W_2}|M_Z)$ given by $\phi(s_1 \otimes s_2)(x) = s_1(x) \otimes s_2(x)$. We note if s_1 and s_2 are parallel then so is $\phi(s_1 \otimes s_2)$.

Thus to prove the theorem we have to prove that the cycle on the right-hand side is not a boundary. For this we need an analogue of Lemma 6.1. The proof is the same as before.

Lemma 7.2 *Assume that $p_1 \leq q_1$, $p_2 \leq q_2$ and $q_1 + q_2 \leq n - (q_1 + q_2)$. Then there exist (infinitely many) $\mathbb{K}$-rational $n - k$-tuples*

$$w = (w_1, \cdots, w_{q_1}, w_{q_1+1}, \cdots, w_{q_1+q_2}, \cdots, w_{n-(q_1+q_2)})$$

in $V^{n-(q_1+q_2)}$ such that if we define $w' = (w_1, \cdots, w_{p_1})$ and $w'' = (w_{q_1+1}, \cdots, w_{q_1+1+p_2})$ then we have

1. *$(\tau_{x'}, \tau_{w'}) \neq 0$.*

2. *$(\tau_{x''}, \tau_{w''}) \neq 0$.*

3. *Let $u = \{e_1, \cdots, e_{q_1+q_2}, w_1, \cdots, w_{n-(q_1+q_2)}\}$ and $U = span\ u$. Then $dim(U) = n$.*

4. *The restriction of $(\ ,\ )$ to U is positive definite.*

We can now prove the Theorem. Put $W = span\ w$. Form the decomposable cycle $M_W \otimes (s_{w'} \otimes s_{w''})$ and intersect with $M_Z \otimes (s_{x'} \otimes s_{y'})$ (after passing to a deep congruence subgroup so that $M_Z \cap M_W$ is a point). We obtain

$$M_Z \otimes (s_{x'} \otimes s_{y'}) \cdot M_W \otimes (s_{w'} \otimes s_{w''}) = (s_{x'}, s_{w'})(s_{y'}, s_{w''}) = (\tau_{x'}, \tau_{w'})(\tau_{y'}, \tau_{w''}) \neq 0.$$

We conclude with an example that shows that $p_1 + p_2 \leq [\frac{n}{2}]$ is not a strong enough assumption to guarantee nonvanishing of the above cup-product.

Example 7.3 *We take $G = SO(6,1), W_1 = \mathbb{R}$ and $W_2 = \bigwedge^3 V$ where V is the standard representation of G. We consider the cup-product*

$$H^1(\Gamma, \mathbb{R}) \otimes H^3(\Gamma, \bigwedge^3 V) \to H^4(\Gamma, \bigwedge^3 V).$$

But by Poincaré duality $H^4(\Gamma, \bigwedge^3 V) \cong H^2(\Gamma, \bigwedge^3 V) = 0$. Note that $p_1 + p_2 = 3 = [\frac{n}{2}]$ whereas $q_1 + q_2 = 4 > [\frac{n}{2}]$.

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